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Exact multi-matrix correlators

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Abstract: We argue that restricted Schur polynomials provide a useful parameterization of the complete set of gauge invariant variables of multi-matrix models. The two point functions of restricted Schur polynomials are evaluated exactly in the free field theory limit. They have diagonal two point functions.

Keywords: 1/N Expansion, AdS-CFT Correspondence, M(atrix) Theories.

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Contents

1. Introduction

The Maldacena conjecture [1], which claims an equivalence between $\mathcal{N} = 4$ super Yang-Mills theory and type IIB string theory in the $\mathrm{AdS}_5\times\mathrm{S}^5$ background, is a significant step in establishing the expectation that large N gauge theories are equivalent to string theory. One approach towards establishing the conjecture is to implement a direct change of variables from the matrices of the gauge theory to the fields of string theory. Collective field theory [2] provides a clear and well defined scheme for making this transition. The first step in this approach is to find a useful parameterization of the complete set of gauge invariant variables of the matrix model. For a model with more than one matrix, this purely kinematical problem is already nontrivial. In this note, we call this the *kinematical problem.*

The $\mathcal{N} = 4$ super Yang-Mills theory has six hermittian Higgs fields, ϕ_i $i = 1, 2, \ldots, 6$, transforming in the adjoint of $\mathrm{U}(N)$. Form the complex combinations $Z = \phi_1 + i\phi_2$, $X = \phi_3 + i\phi_4$ and $Y = \phi_5 + i\phi_6$. The space of $\frac{1}{2}$ BPS representations in $\mathcal{N} = 4$ super Yang-Mills theory are in one-to-one correspondence with the Schur polynomials built out of Z [3]. These Schur polynomials have diagonal two point functions [3]. Using insights from the dual quantum gravity, excitations of these $\frac{1}{2}$ BPS states, *restricted Schur polynomials*, have been identified [4]. The restricted Schur polynomial is obtained by "attaching" open string words W to the Schur polynomial. The letters of these open string words can be fermions, gauge fields or any of the six Higgs fields. If the word W is to be dual to an open string, it should contain $O(\sqrt{N})$ letters. If the restricted Schur polynomial contains $O(N)$ fields, it is dual to a membrane with open strings attached; if it contains $O(N^2)$ fields, it is dual to a string moving in a new geometry. Thus, the restricted Schur polynomial starts to address the kinematical problem outlined above. The technology for computing correlators of restricted Schur polynomials has enjoyed some progress [5 − 7]. For related work see [8].

In two recent papers, a large class of operators that diagonalize the two point functions in the free field theory limit have been given [9, 10]. These include operators built from Z and $Z^\dagger$ [9] and operators built from X, Y and Z [10]. Further, the number of such operators matches the number of gauge invariant operators that can be constructed. The results of [9, 10] therefore solve the kinematical problem, in the Higgs sector. This basis also gives a group theoretic way to approach higher point functions (see [10] where three and higher point functions are obtained) and to obtain factorization equations which can be used to build a probability interpretation [11]. By exploiting supergroups [10] have also explained how to include fermions in addition to the Higgs fields. Finally, the one loop correction to these two points functions has been considered in [12].

The purpose of this communication is to argue that the restricted Schur polynomials themselves, provide a solution to the kinematical problem, in the Higgs sector. This is not unexpected. Indeed, if one excites a $\frac{1}{2}$ BPS state by attaching a large number of words that are composed of a single letter, one is building up multi-matrix operators. Our argument is simple, employing only very basic group theory. Further, by exploiting the technology already available for restricted Schur polynomials, explicit formulas for the relevant restricted Schur polynomials and their two point functions are easily obtained.

2. Two matrix model

Consider a $d = 0$ matrix model with two types of complex matrices A and B.[1] These complex matrices act on an N-dimensional vector space V, $A : V \to V$. The non-zero correlators are

$$\langle (A)^i_j (A^\dagger)^k_l \rangle = \delta^i_l \delta^k_j = \langle (B)^i_j (B^\dagger)^k_l \rangle. \tag{2.1}$$

Consider the operators

$$\chi_\alpha = \mathrm{Tr}_{\,n+m}(O_\alpha A^{\otimes n} \otimes B^{\otimes m}),$$

where $\mathrm{Tr}_{\,n+m}$ is a trace over $V^{\otimes(n+m)}$. $A^{\otimes n} \otimes B^{\otimes m}$ is a shorthand for the tensor $A^{i_1}_{j_1} A^{i_2}_{j_2} \cdots A^{i_n}_{j_n} B^{i_{n+1}}_{j_{n+1}} B^{i_{n+2}}_{j_{n+2}} \cdots B^{i_{n+m}}_{j_{n+m}}$ and,

$$\mathrm{Tr}_{\,n+m}(O_\alpha A^{\otimes n} \otimes B^{\otimes m}) = (O_\alpha)^{j_1 j_2 \cdots j_{n+m}}_{i_1 i_2 \cdots i_{n+m}} A^{i_1}_{j_1} A^{i_2}_{j_2} \cdots A^{i_n}_{j_n} B^{i_{n+1}}_{j_{n+1}} B^{i_{n+2}}_{j_{n+2}} \cdots B^{i_{n+m}}_{j_{n+m}}.$$

We are interested in computing the correlator $\langle \chi_\alpha \chi^\dagger_\beta \rangle$. Using (2.1) we obtain

$$\langle \chi_\alpha \chi^\dagger_\beta \rangle = \sum_{\gamma \in S_n \times S_m} \mathrm{Tr}_{\,n+m}(O_\alpha \gamma O^\dagger_\beta \gamma^{-1}).$$

The sum over γ is a sum over all possible Wick contractions. Assume that

$$O_\beta = \gamma O_\beta \gamma^{-1}, \qquad n!m!\,\mathrm{Tr}_{\,n+m}(O_\alpha O^\dagger_\beta) = \mathcal{N}_\alpha \delta_{\alpha\beta}.$$

[1] The spacetime dependence which has been dropped from this model can be trivially reinstated using the conformal symmetry of the super Yang-Mills theory.

This means that the O_α are symmetric branching operators [9]. Then the operators χ_α diagonalize the two point function

$$\langle \chi_\alpha \chi_\beta^\dagger \rangle = \mathcal{N}_\alpha \delta_{\alpha\beta}.$$

We will now argue that a complete set of O_α are given by

$$O_\alpha = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))\sigma,$$

where R_α is an irreducible representation of $S_n \times S_m$ and R is an irreducible representation of S_{n+m}. The $S_n \times S_m$ subgroup is chosen so that S_n acts on the indices of the As and S_m on the indices of the Bs. Thus, the $S_n \times S_m$ subgroup that we sum over to include all possible Wick contractions is the same subgroup for which R_α is an irreducible representation. Under restricting to the $S_n \times S_m$ subgroup, R will in general be reducible. We can decompose the carrier space of irreducible representation R according to the irreducible $S_n \times S_m$ representations that are subduced. Tr_{R_α} is an instruction to trace only over the subspace corresponding to R_α. For more details see [5]. In this case, the χ_α are nothing but restricted Schur polynomials, so that the restricted Schur polynomials solve the kinematical problem and have diagonal two point functions.

Demonstration that $\boldsymbol{O_\beta = \gamma O_\beta \gamma^{-1}}$:

$$\begin{aligned}
\gamma O_\alpha \gamma^{-1} &= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))\gamma\sigma\gamma^{-1} = \frac{1}{n!m!} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\gamma^{-1}\tau\gamma))\tau \\
&= \frac{1}{n!m!} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\gamma^{-1})\Gamma_R(\tau)\Gamma_R(\gamma))\tau \\
&= \frac{1}{n!m!} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_{R_\alpha}(\gamma^{-1})\Gamma_R(\tau)\Gamma_{R_\alpha}(\gamma))\tau \\
&= \frac{1}{n!m!} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\tau))\tau = O_\alpha.
\end{aligned}$$

We used the fact that $\gamma \in S_n \times S_m$, that R_α is an irreducible representation of $S_n \times S_m$ and that the trace is invariant under a similarity transformation.

Demonstration that $\boldsymbol{\mathrm{Tr}_{n+m}(O_\alpha O_\beta^\dagger) = \mathcal{N}_\alpha \delta_{\alpha\beta}}$:

$$\begin{aligned}
n!m!\mathrm{Tr}_{n+m}(O_\alpha O_\beta^\dagger) &= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))\mathrm{Tr}_{S_\beta}(\Gamma_S(\tau))^* \mathrm{Tr}_{n+m}(\sigma\tau^{-1}) \\
&= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))\mathrm{Tr}_{S_\beta}(\Gamma_S(\tau))^* N^{C(\sigma\tau^{-1})} \\
&= \frac{1}{n!m!} \sum_{\psi \in S_{n+m}} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi\tau))\mathrm{Tr}_{S_\beta}(\Gamma_S(\tau))^* N^{C(\psi)}.
\end{aligned}$$

Now, lets perform the sum over τ (use the fact that $(P_{S\to S_\beta})^*_{jr} = (P_{S\to S_\beta})_{rj}$ because the projector $P_{S\to S_\beta}$ is hermittian)

$$\begin{aligned}
&\sum_{\tau\in S_{n+m}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi\tau))\mathrm{Tr}_{S_\beta}(\Gamma_S(\tau))^* \\
&\quad= \sum_{\tau\in S_{n+m}}\sum_{i\,j\,q\,r}(P_{R\to R_\alpha}\Gamma_R(\psi))_{iq}(\Gamma_R(\tau))_{qi}(P_{S\to S_\beta})_{rj}(\Gamma_S(\tau))^*_{rj} \\
&\quad= \delta_{RS}\frac{(n+m)!}{d_R}\sum_{i\,q}(P_{R\to R_\alpha}\Gamma_R(\psi))_{iq}(P_{S\to S_\beta})_{qi} = \delta_{RS}\delta_{R_\alpha S_\beta}\mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi))\frac{(n+m)!}{d_R}.
\end{aligned}$$

We have used the fundamental orthogonality relation

$$\sum_{\tau\in S_{n+m}}(\Gamma_R(\tau))_{qi}(\Gamma_S(\tau))^*_{rj} = \frac{(n+m)!}{d_R}\delta_{qr}\delta_{ij}\delta_{RS}.$$

Thus, using appendix F of [5] we obtain

$$\begin{aligned}
n!m!\mathrm{Tr}\,(O_\alpha O_\beta^\dagger) &= \frac{\delta_{RS}\delta_{R_\alpha S_\beta}}{n!m!}\frac{(n+m)!}{d_R}\sum_{\psi\in S_{n+m}}\mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi))N^{C(\psi)} \\
&= \frac{\delta_{RS}\delta_{R_\alpha S_\beta}}{n!m!}\frac{(n+m)!}{d_R}d_{R_\alpha}f_R = \delta_{RS}\delta_{R_\alpha S_\beta}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R\,.
\end{aligned}$$

R_α is a rep of $S_n\times S_m$ which is labelled by one Young diagram of n boxes, R_n, and one Young diagram of m boxes, R_m. $(\mathrm{hooks})_{R_\alpha}$ is the product of $(\mathrm{hooks})_{R_n}$ with $(\mathrm{hooks})_{R_m}$. Arguing as we did above, it is simple to obtain

$$O_\alpha O_\beta = \frac{(n+m)!}{d_R n!m!}\delta_{\alpha\beta}O_\alpha.$$

Thus, up to normalization our operators O_α are projectors. For an earlier use of projectors, along the lines of this note but in the setting of a single matrix, see [13]. From now on we write χ_{R,R_α} instead of χ_α. In general, the row and column index of the restriction R_α can be different (see [4, 5] for a detailed discussion). Spell out these row and column indices by replacing $R_\alpha\to(r_{\alpha_1},r_{\alpha_2})$. The two point function is

$$\langle\chi_{R,(r_{\alpha_1},r_{\alpha_2})}\chi^\dagger_{S,(s_{\beta_1},s_{\beta_2})}\rangle = \delta_{RS}\delta_{r_{\alpha_1}s_{\beta_1}}\delta_{r_{\alpha_2}s_{\beta_2}}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R\,. \tag{2.2}$$

It is equally easy to argue that

$$\langle\chi_{R,(r_{\alpha_1},r_{\alpha_2})}\chi_{S,(s_{\beta_1},s_{\beta_2})}\rangle = \delta_{RS}\delta_{r_{\alpha_1}s_{\beta_2}}\delta_{r_{\alpha_2}s_{\beta_1}}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R\,. \tag{2.3}$$

In sections 3 and 5 we will give evidence that the number of restricted Schur polynomials χ_{R,R_α} is equal to the number of gauge invariant operators in the matrix model.

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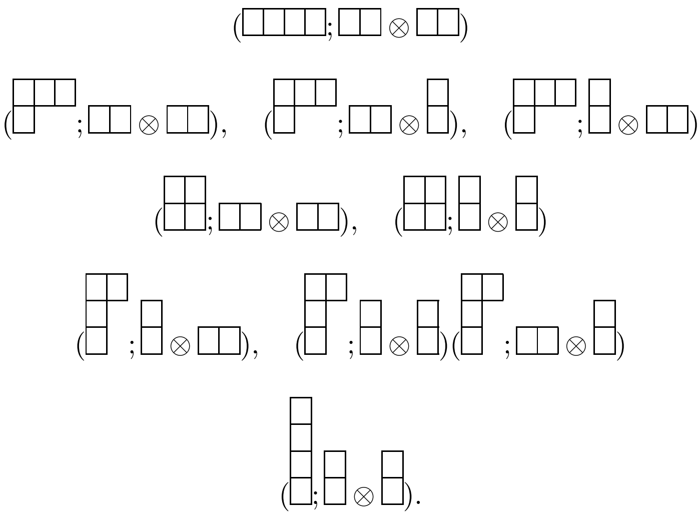
3. Counting

The number of gauge invariant operators $N(n,m)$ built out of n As and m Bs is given by Polya theory as

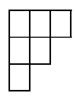
$$\prod_{k=1}^{\infty} \frac{1}{1-(x^k+y^k)} = \sum_{n,m} N(n,m) x^n y^m.$$

We claim that the number of gauge invariant operators $N(n,m)$ is equal to the number of restricted Schur polynomials χ_{R,R_α} with R an irreducible representation of S_{n+m} and R_α an irreducible representation of $S_n \times S_m$. It is easy to check for small values of n and m that this is indeed the case. As an example, consider $m = n = 2$. In this case, R is an irreducible representation of S_4. We easily find $N(2,2) = 10$. The allowed restricted traces $(R; R_\alpha)$ are

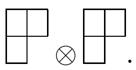


Thus, there are indeed ten possible restricted Schur polynomials.

There is a subtlety that did not show up in the above example: in the notation of [5, 4], we can trace over an off the diagonal block. For example, amoung the $S_3 \times S_3$ irreducible representations subduced by the S_6 irreducible representation

$R =$ 

we find two copies of



Call these two copies $R_\alpha^{(1)}$ and $R_\alpha^{(2)}$. When performing the restricted trace, we can use $R_\alpha^{(i)}$ for the row index and $R_\alpha^{(j)}$ for the column index with $R_\alpha^{(i)} \neq R_\alpha^{(j)}$. Thus, there are four possible operators we can define. In general, if R subduced m copies of an irreducible representation R_α we would be able to construct m^2 independent operators. For further details consult section 2.2 of [5].

4. Examples

The simplest way to construct restricted Schur polynomials, is to use a projection operator to implement the restricted trace. In this section we will construct restricted Schur polynomials built from at most three matrices, which can be any of two different types X or Y. This will already allow us to see that the restricted Schur polynomials define a different basis for gauge invariant operators, than the bases given in [9, 10]. The construction of

$$\chi_{\yng(2);\yng(1)\otimes\yng(1)} = \mathrm{Tr}\,(X)\mathrm{Tr}\,(Y) + \mathrm{Tr}\,(XY), \qquad \chi_{\yng(1,1);\yng(1)\otimes\yng(1)} = \mathrm{Tr}\,(X)\mathrm{Tr}\,(Y) - \mathrm{Tr}\,(XY),$$

is particularly simple because we do not need a projector to implement the restricted trace. This follows because $\yng(1)\otimes\yng(1)$ is the only $S_1 \times S_1$ irreducible representation subduced from either $\yng(2)$ or $\yng(1,1)$. Up to normalization, these are identical to the operators constructed in appendix E1 of [10]. Consider next

$$\chi_{\yng(3);\yng(2)\otimes\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}\,(X)^2\mathrm{Tr}\,(Y) + \mathrm{Tr}\,(X^2)\mathrm{Tr}\,(Y) + 2\mathrm{Tr}\,(XY)\mathrm{Tr}\,(X) + 2\mathrm{Tr}\,(X^2Y)\right],$$

$$\chi_{\yng(1,1,1);\yng(1,1)\otimes\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}\,(X)^2\mathrm{Tr}\,(Y) - \mathrm{Tr}\,(X^2)\mathrm{Tr}\,(Y) - 2\mathrm{Tr}\,(XY)\mathrm{Tr}\,(X) + 2\mathrm{Tr}\,(X^2Y)\right].$$

For these two restricted Schur polynomials we again do not need a projector to implement the restricted trace. If we take

$$\chi_{R,R_\alpha} = \frac{1}{2!}\sum_{\sigma\in S_3} \mathrm{Tr}\,_{R_\alpha}(\Gamma_R(\sigma))X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}Y^{i_3}_{i_{\sigma(3)}},$$

then R_α is an irreducible representation of $S_2\times S_1$. The S_2 subgroup is obtained by taking those elements of S_3 that act on the indices of the Xs, i.e. $\{1,(12)\}$. To compute

$$\chi_{\yng(2,1);\yng(2)\otimes\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}\,(X)^2\mathrm{Tr}\,(Y) + \mathrm{Tr}\,(X^2)\mathrm{Tr}\,(Y) - \mathrm{Tr}\,(XY)\mathrm{Tr}\,(X) - \mathrm{Tr}\,(X^2Y)\right],$$

we used the projector

$$P_{\yng(2,1)\to\yng(2)\otimes\yng(1)} = \frac{1}{2}\left(1 + \Gamma_{\yng(2,1)}((12))\right).$$

To compute

$$\chi_{\yng(2,1);\yng(1,1)\otimes\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}\,(X)^2\mathrm{Tr}\,(Y) - \mathrm{Tr}\,(X^2)\mathrm{Tr}\,(Y) + \mathrm{Tr}\,(XY)\mathrm{Tr}\,(X) - \mathrm{Tr}\,(X^2Y)\right],$$

we used the projector

$$P_{\yng(2,1)\to\yng(1,1)\otimes\yng(1)} = \frac{1}{2}\left(1 - \Gamma_{\yng(2,1)}((12))\right).$$

For more details on these projectors see appendix B.2 of [5] and appendix A of [7]. Comparing these expressions to the expressions in appendix E.2 of [10], it is clear that the basis furnished by the restricted Schur polynomials does not coincide with the basis of [10].

We can use the Σ map of [9] to construct new operators built out of Z and Z^*. Under the map Σ, $B_\alpha = \Sigma^{-1}(O_\alpha)$ becomes a sum over elements of the Brauer algebra. In [9] it was argued that if $\gamma O_\alpha \gamma^{-1} = O_\alpha$ for $\gamma \in S_n \times S_m$ then $\gamma B_\alpha \gamma^{-1} = B_\alpha$ for $\gamma \in S_n \times S_m$. Also, again using a result of [9], ($\mathrm{Tr}_{\,m+n}$ denotes a trace over $V^{\otimes(n+m)}$ and $\mathrm{Tr}_{\,m,n}$ denotes a trace over $V^{\otimes n} \otimes \bar{V}^{\otimes m}$)

$$\mathrm{Tr}_{\,m,n}(B_\alpha B_\beta) = \mathrm{Tr}_{\,m+n}(O_\alpha O_\beta) = \frac{\mathcal{N}_\alpha}{n!m!}\delta_{\alpha\beta}.$$

Thus, the operators

$$\eta_\alpha = \mathrm{Tr}_{\,m,n}(B_\alpha Z^{\otimes n} \otimes Z^{*\,\otimes m}),$$

have a diagonal two point function

$$\langle \eta_\alpha \eta_\beta^\dagger \rangle = \mathcal{N}_\alpha \delta_{\alpha\beta}.$$

For $m = n = 1$ we find

$$B_1 = \Sigma\left(\frac{1}{2}(1 + (12))\right) = \frac{1}{2}\left(1 + C_{1\bar{1}}\right),$$

$$B_2 = \Sigma\left(\frac{1}{2}(1 - (12))\right) = \frac{1}{2}\left(1 - C_{1\bar{1}}\right).$$

These do not match the operators given in appendix A.4.1 of [9], implying that the restricted Schur polynomials do not coincide with the basis constructed in [9] either. This is clear when one notes that the coefficients on the projectors in [9] are N dependent; there is no way in which our operators could pick up N dependent coefficients.

Recall that weights are assigned to boxes in a Young diagram by assigning N to the box in the upper left hand corner of the Young diagram, adding one each time we move to the right and subtracting one each time we move down. Thus, box i in the Young diagram

$$\young(123,45)$$

has weight c_i with $c_1 = c_5 = N$, $c_2 = N + 1$, $c_3 = N + 2$ and $c_4 = N - 1$. f_R is the product of weights of the Young diagram, so that, for example

$$f_{\yng(3,2)} = N^2(N^2 - 1)(N + 2).$$

Next, since

$$\mathrm{hooks}(\yng(5)) = 5! \quad \mathrm{hooks}(\yng(3)) = 3! \quad \mathrm{hooks}(\yng(2)) = 2!,$$

we have from (2.2)

$$\langle \chi_{\yng(5);\yng(3)\otimes\yng(2)} \chi^\dagger_{\yng(5);\yng(3)\otimes\yng(2)} \rangle = \frac{5!}{3! \times 2!} f_{\yng(5)}.$$

Similary,

$$\langle \chi_{\yng(3,3);\yng(3)\otimes\yng(3)} \chi^\dagger_{\yng(3,3);\yng(3)\otimes\yng(3)} \rangle = \frac{4! \times 3!}{3! \times 3!} f_{\yng(3,3)}.$$

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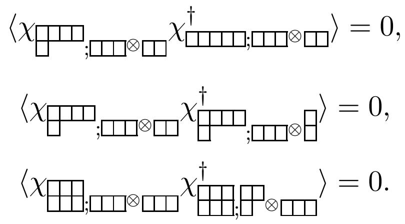
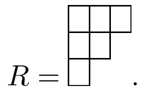
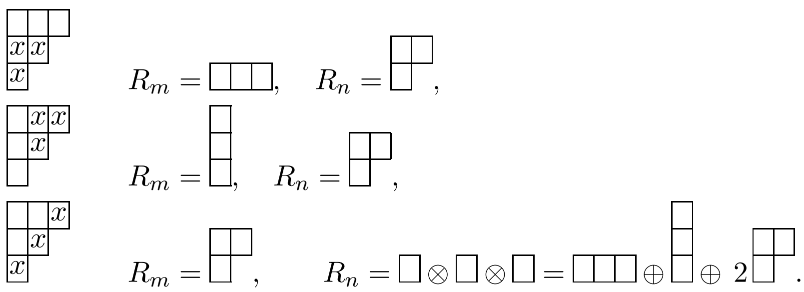
If any of the labels on the restricted Schur polynomial do not match, the correlator vanishes



To determine which $S_n \times S_m$ irreducible representations are subduced by a particular S_{n+m} irreducibe representation is easy: assume that the Young diagram R describes the irreducible representation of S_{n+m} that we are studying. Consider all possible ways of removing n boxes from R so that the remaining m boxes form a legal Young diagram R_m. Remove the n boxes preserving common sides and take the tensor product of the removed pieces to get R_n. This rule is easily illustrated with an example; consider



Assume that $n = m = 3$. Denoting removed boxes with an x we have



Thus, R subduces 6 irreducible representations of $S_3 \times S_3$.

5. Generalization to multi-matrix models

The above results generalize in a simple way to multi-matrix models. Consider a model of M matrices and assume that χ_{R,R_α} is built from m_i matrices of each type. Then R_α is an irreducible representation of $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_M}$. To remove self contractions (present if we have real matrices or if we build χ_{R,R_α} from complex matrices and their adjoints) we simply normal order χ_{R,R_α}. This gives a unified treatment of both branes/antibrane systems and operators built from more than one Higgs field. These operators are labeled by $M+1$ Young diagrams, one with $m_1 + m_2 + \ldots + m_M$ boxes, R and M with m_i boxes, R_i. In this more general case we still have (2.2) with

$$(\text{hooks})_{R_\alpha} = \prod_{i=1}^{M} (\text{hooks})_{R_i}.$$

It is straight forward to replace boxes in the R_i by open strings so that excited operators can be constructed and studied using the techniques developed in [5–7].

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We again claim that the total number of restricted Schur polynomials that can be defined will be equal to the number of gauge invariant operators that can be constructed. There are some non-trivial tests we can perform of this claim. For example, consider operators built using one of each of the M types. In this case, we need to start with an irreducible representation of S_M and count how many restricted Schur polynomials we can form when the representation of the restriction is $S_1 \times S_1 \times \cdots \times S_1$ (there are M factors). To get the number of irreducible representations that can be subduced from a given Young diagram R, we need to count the number of ways we can pull boxes off R such that at each step we have a legal Young diagram. This is obviously d_R, the dimension of the S_M representation labeled by R. Any of these subduced representations may be twisted, so that we obtain a total of d_R^2 operators. Thus, the total number of restricted Schur polynomials, found by summing over all S_M irreducible representations, is simply

$$\sum_R (d_R)^2 = M!.$$

Lets now compare this to the counting of the gauge invariant operators. According to Polya theory, the number of gauge invariant operators is given by the coefficient of $x_1 x_2 \cdots x_M$ in the expansion of

$$\prod_{k=1}^{\infty} \frac{1}{1-(x_1^k + x_2^k + \cdots + x_M^k)} = \sum_{n_1, n_2, \cdots, n_m} t(n_1, n_2, \cdots, n_M) x_1^{n_1} x_2^{n_2} \cdots x_M^{n_M}.$$

It is simple to see that

$$t(1,1,\cdots,1) = M!,$$

which supports our claim.

6. Numerical tests

We have counted the number of restricted Schur polynomials χ_{R,R_α} that be obtained when R is an irreducible representation of S_n with $n \le 6$ and we have a total of $M = 6$ matrices. In all of these cases, the number of restricted Schur polynomials equals the number of gauge invariant operators counted using Polya theory. Further, we have numerically evaluated the two point functions of these restricted Schur polynomials and verified that (2.2) is indeed correct. In performing these checks, the restricted characters $\mathrm{Tr}_{R_\alpha}(\Gamma_R[\sigma])$ were evaluated by explicitly constructing the matrices $\Gamma_R[\sigma]$. Each representation used was obtained by induction. One induces a reducible representation; the irreducible representation required was isolated using projection operators built from the Casimir obtained by summing over all two cycles. The restricted trace was then evaluated with the help of suitable projectors. See appendix B.2 of [5] and appendix A of [7] for more details. In all cases the numerical result is in exact agreement with (2.2).

7. Conclusions

Restricted Schur polynomials provide a useful parameterization of the complete set of gauge invariant variables of multi-matrix models. They have diagonal two point functions. Since in the labeling of the restricted Schur polynomial, each type of matrix has its own Young diagram, the technology for attaching open strings has a straight forward generalization to the operators considered in this article.

For brane-anti-brane operators, the restricted Schur polynomials do not coincide with the Brauer basis constructed in [9]. Since the Brauer projectors are N dependent, the relation between the two bases is N dependent. It seems that the Brauer basis may be the most useful for identifying brane - anti-brane operators and the restricted Schur polynomial basis for stringy excitations. It is plausible that there is a simple relation between the restricted Schur polynomials and the operators of [10]. For example, $\chi_{\yng(2,1);\yng(2)\otimes\yng(1)} - \chi_{\yng(2,1);\yng(1,1)\otimes\yng(1)}$ is (up to an overall constant factor) equal to the operators constructed in E.2 and E.3 of [10]. Since the restricted Schur polynomials have an interpretation in terms of attaching open strings, developing this relation may well shed light on the interpretation of the labels of the operators constructed in [10]. We leave this interesting problem for the future.

Finally, it would be interesting to explore finite N effects. These effects are encoded in the fact that our Young diagram labels can have at most N rows. Specifically, in the restricted Schur polynomial $\chi_{R,(r_{\alpha_1},r_{\alpha_2})}$ we must require that the Young diagram R has at most N rows; the same will automatically be true for r_{α_1} and r_{α_2}. This should translate into a generalization of the stringy exclusion principle present for Schur polynomials built using a single matrix Z. Finite N counting for multi-matrix operators has been considered in [10, 14]. For example, the number of operators built using μ_1 X fields and μ_2 Y fields, at infinite N is given by[2]

$$N(\mu_1,\mu_2) = \sum_T \sum_\Lambda C(T,T,\Lambda) g(\mu;\Lambda).$$

In this formula, T is a representation of S_n with $n = \mu_1 + \mu_2$, $C(T,T,\Lambda)$ is the coefficient of Λ in the (inner) tensor product $T \otimes T$ and $g(\mu;\Lambda)$ is the Littlewood-Richardson coefficient which counts states in the representation Λ that have the field content $\mu = [\mu_1] \otimes [\mu_2]$. To get the finite N counting, one simply truncates the sum over T to Young diagrams with at most N rows. We can see, in some simple examples, that our cut off on R does indeed match the finite N counting of [10, 14]. Consider for example the operators built using 3 X fields and a single Y field. The relevant Littlewood-Richardson coefficients are

$$g(\yng(3),\yng(1);\yng(4)) = 1, \qquad g(\yng(3),\yng(1);\yng(3,1)) = 1.$$

The relevant inner products are

$$\yng(4) \otimes \yng(4) = \yng(4),$$

$$\yng(3,1) \otimes \yng(3,1) = \yng(4) \oplus \yng(3,1) \oplus \yng(2,2) \oplus \yng(2,1,1),$$

$$\yng(2,2) \otimes \yng(2,2) = \yng(4) \oplus \yng(2,2) \oplus \yng(1,1,1,1),$$

$$\yng(2,1,1) \otimes \yng(2,1,1) = \yng(4) \oplus \yng(3,1) \oplus \yng(2,2) \oplus \yng(2,1,1),$$

[2] We are considering the case of two matrices for simplicity. The formula for M matrices has been determined in [10, 14].

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$$\yng(1,1,1,1) \otimes \yng(1,1,1,1) = \yng(4).$$

Clearly then, at infinite N, the number of operators we can construct is

$$\begin{aligned} N(2,1) &= \sum_T \left(C(T,T,\yng(4)) + C(T,T,\yng(3,1)) \right) \\ &= C(\yng(4),\yng(4),\yng(4)) + C(\yng(3,1),\yng(3,1),\yng(4)) + C(\yng(2,2),\yng(2,2),\yng(4)) + C(\yng(2,1,1),\yng(2,1,1),\yng(4)) \\ &\quad + C(\yng(1,1,1,1),\yng(1,1,1,1),\yng(4)) + C(\yng(3,1),\yng(3,1),\yng(3,1)) + C(\yng(2,1,1),\yng(2,1,1),\yng(3,1)) = 7. \end{aligned}$$

At $N = 2$, this counting becomes

$$\begin{aligned} N(2,1) &= \sum_T \left(C(T,T,\yng(4)) + C(T,T,\yng(3,1)) \right) \\ &= C(\yng(4),\yng(4),\yng(4)) + C(\yng(3,1),\yng(3,1),\yng(4)) + C(\yng(2,2),\yng(2,2),\yng(4)) \\ &\quad + C(\yng(3,1),\yng(3,1),\yng(3,1)) = 4. \end{aligned}$$

Lets now count the restricted Schur polynomials. At infinite N we find 7 possible operator, with $R, r_{\alpha_1} r_{\alpha_2}$ labels given by

$$\yng(4),\yng(3)\,\yng(1) \qquad \yng(3,1),\yng(3)\,\yng(1) \qquad \yng(3,1),\yng(2,1)\,\yng(1) \qquad \yng(2,2),\yng(2,1)\,\yng(1)$$

$$\yng(2,1,1),\yng(2,1)\,\yng(1) \qquad \yng(2,1,1),\yng(1,1,1)\,\yng(1) \qquad \yng(1,1,1,1),\yng(1,1,1)\,\yng(1)$$

At $N = 2$ there are only 4 operators, with labels given by

$$\yng(4),\yng(3)\,\yng(1) \qquad \yng(3,1),\yng(3)\,\yng(1) \qquad \yng(3,1),\yng(2,1)\,\yng(1) \qquad \yng(2,2),\yng(2,1)\,\yng(1)$$

Providing a proof that our cut off on R matches the finite N counting of [10, 14] remains an interesting open problem.

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